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Recollements of extension algebras

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Abstract Let A be a finite-dimensional algebra over arbitrary base field k. We prove: if the unbounded derived module category $D^{-}(\mathsf{Mod}\text{-}A)$ admits symmetric recollement relative to unbounded derived module categories of two finite-dimensional k-algebras B and C:

$$D^{-}(\mathsf{Mod}-B) \overset{\leftarrow}{\rightleftarrows} D^{-}(\mathsf{Mod}-A) \overset{\leftarrow}{\rightleftarrows} D^{-}(\mathsf{Mod}-C),$$

then the unbounded derived module category $D^{-}(\mathsf{Mod}-T(A))$ admits symmetric recollement relative to the unbounded derived module categories of $T(B)$ and $T(C)$:

$$D^{-}(\mathsf{Mod}-T(B)) \overset{\leftarrow}{\rightleftarrows} D^{-}(\mathsf{Mod}-T(A)) \overset{\leftarrow}{\rightleftarrows} D^{-}(\mathsf{Mod}-T(C)).$$

Keywords: trivial extension algebras, derived categories, (symmetric) recollements, partial tilting complexes.

1 Main result

As we all know, the tilting theory and the theory of derived categories are two important research subjects in the representation theory of finite-dimensional algebras, and many advanced results have been obtained in the last decades (see, for example, refs. [1—4]).

Originally, recollement for derived categories was introduced by Grothendieck in order to describe the relation between the sheaves on a topological ringed space X and those sheaves on X induced from a closed subspace and its open complement respectively[5]. Cline et al. used the definition of recollement of triangulated categories given by Beilinson et al. in their work on perverse sheaves as a generalization of ref. [6] to obtain what they call stratification of derived module categories of certain algebras (for example, quasi-hereditary algebras)[7]. Many mathematicians[7−10] studied recollements of triangulated categories.

Definition[6]**.** Let$\mathcal{D}, \mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories. Then a recollement of $\mathcal{D}$ relative to $\mathcal{D}'$ and $\mathcal{D}''$, diagrammatically expressed by

$$\mathcal{D}' \overset{\leftarrow}{\rightleftarrows} \mathcal{D} \overset{\leftarrow}{\rightleftarrows} \mathcal{D}'',$$

is given by six exact functors

$$i_\star = i_! : \mathcal{D}' \to \mathcal{D}; \quad j^\star = j^! : \mathcal{D} \to \mathcal{D}''; \quad i^\star, i^! : \mathcal{D} \to \mathcal{D}'; \quad j_\star, j_! : \mathcal{D}'' \to \mathcal{D},$$

which satisfy the following four conditions:

(R1) $(i^\star, i_\star = i_!, i^!)$ and $(j_!, j^\star = j^!, j_\star)$ are adjoint triples, i.e. $i^\star$ is left adjoint to $i_\star$ which is left adjoint to $i^!$, etc.;

(R2) $i^! j_\star = 0$ (and thus $j^\star i_\star = 0$ and $i^\star j_! = 0$);

(R3) $i_\star$, $j_!$ and $j_\star$ are full embeddings (and thus $i^\star i_\star \cong i^! i_\star \cong id(\mathcal{D}')$ and $j^\star j_\star \cong j^\star j_! \cong id(\mathcal{D}''))$;

(R4) any object X in $\mathcal{D}$ determines distinguished triangles

$$i_! i^! X \to X \to j_\star j^\star X \to \quad \text{and} \quad j_! j^! X \to X \to i_\star i^\star X \to,$$

where the morphisms $i_! i^! X \to X$ and $j_! j^! X \to X$ are the front adjunction morphisms; $X \to j_\star j^\star X$ and $X \to i_\star i^\star$are the rear adjunction morphisms.

Throughout this paper, all algebras are finite-dimensional algebras over arbitrary base field k. Let DA be minimal injective cogenerators of mod$A$, where $D = \mathrm{Hom}_k(-, k)$ is the duality functor. Let $T(A)$ denote the trivial extension algebra of A. Then the trivial extension algebra $T(A)$ of A by DA is defined as the k-algebra whose additive structure is that of $A \oplus DA$ and whose multiplication structure is given by

$$(a, q)(a', q') = (aa', aq' + qa')$$

for $a, a' \in A$ and $q, q' \in DA$.

Unless stated otherwise, all our modules will be right modules. Given a k-algebra A, there are various categories of A-modules for which it will be useful to fix notation. Mod-A is the category of all A-modules; mod-A is the category of all finitelly generated A-modules; by $D^-(\mathrm{Mod} - A)$ we mean the derived category of unbounded complexes over Mod-A; by $D^b(A)$ we mean the derived category of bounded complexes over mod-A; Proj-A is the category of all projective A-modules; P_A is the category of all finitelly generated projective A-modules; by $K^b(\text{Proj-}A)$ we mean the homotopy category of bounded complexes over Proj-A; by $K^b(P_A)$ we mean the homotopy category of bounded complexes over P_A. Since for each algebra A there is full embedding between the various derived categories and homotopy category associated to A, we may denote homomorphisms in any of these categories just by $\mathrm{Hom}_A(-, -)$. We shall denote by $M[n]$ rather than $T^n M$ the object obtained from a complex M by applying the "shift" functor n times. For unexplained notations, properties concerning derived categories and triangulated categories, refer to refs. [1—3], etc.

There are many important results on the recollements of triangulated categories. For convenience, we give an important Koenig's result as follows[9].

Lemma A. Let A be a ring. Then the unbounded derived module category $D^-(\mathrm{Mod}-A)$ admits recollement relative to the unbounded derived module categories of two rings B and C:

$$D^-(\mathrm{Mod} - B) \overset{\leftarrow}{\underset{\leftarrow}{\rightarrow}} D^-(\mathrm{Mod} - A) \overset{\leftarrow}{\underset{\leftarrow}{\rightarrow}} D^-(\mathrm{Mod} - C)$$

if and only if there exist two partial-tilting complexes $M^\cdot \in K^b(\mathrm{Proj} - A)$ and $N^\cdot \in K^b(P_A)$ that satisfy

(1) $\mathrm{End}_A(M^\cdot) \cong B$;

(2) $\mathrm{End}_A(N^\cdot) \cong C$;

(3) $\mathrm{Hom}^\cdot_A(N^\cdot, M^\cdot) = 0$ (i.e. $\mathrm{Hom}_A(N^\cdot, M^\cdot[n]) = 0$ for any $n \in \mathbb{Z}$);

(4) $(M^{\cdot})^{\perp} \cap (N^{\cdot})^{\perp} = 0$ (where $(M^{\cdot})^{\perp} = \{Y^{\cdot} \in D^{-}(\mathrm{Mod}-A) | \mathrm{Hom}^{\cdot}_{A}(M^{\cdot}, Y^{\cdot}) = 0\}$ is called the right perpendicular category determined by $M^{\cdot}$).

We say that the recollement in Lemma A is symmetric, if it satisfies the additional conditions: $\mathrm{Hom}^{\cdot}_{A}(M^{\cdot}, N^{\cdot}) = 0$ and $M^{\cdot} \in K^{b}(P_A)$. In this case, one has

$$D^{-}(\mathrm{Mod}-C) \overset{\leftarrow}{\rightleftharpoons} D^{-}(\mathrm{Mod}-A) \overset{\leftarrow}{\rightleftharpoons} D^{-}(\mathrm{Mod}-B).$$

On symmetric recollement, Koenig has given some interesting characterizations[9].

Furthermore, if $gld.A < \infty$ or $gld.C < \infty$, then $D^{-}(\mathrm{Mod}-A)$ admits the recollement

$$D^{-}(\mathrm{Mod}-B) \overset{\leftarrow}{\rightleftharpoons} D^{-}(\mathrm{Mod}-A) \overset{\leftarrow}{\rightleftharpoons} D^{-}(\mathrm{Mod}-C)$$

if and only if $D^{b}(\mathrm{Mod}-A)$ admits recollement

$$D^{b}(\mathrm{Mod}-B) \overset{\leftarrow}{\rightleftharpoons} D^{b}(\mathrm{Mod}-A) \overset{\leftarrow}{\rightleftharpoons} D^{b}(\mathrm{Mod}-C).$$

As we know, a partial tilting complex over a ring A is a complex $T^{\cdot}$ in $K^{b}(\mathrm{Proj}-A)$ and satisfies

(1) $\mathrm{Hom}_{A}(T^{\cdot}, T^{\cdot}[n]) = 0$ for $n \neq 0$; and

(2) for all indexed families $\{T^{\cdot}_{i}\}_{i \in I}$ of copies of T holds:

$$\oplus_{i\in I}\mathrm{Hom}_{A}(T^{\cdot}, T^{\cdot}_{i}) \overset{\text{nat.}}{\cong} \mathrm{Hom}_{A}(T^{\cdot}, \oplus_{i\in I} T^{\cdot}_{i}).$$

It is easy to see that partial tilting complexes are generalizations of tilting complexes and tilting complexes are generalizations of tilting modules[2].

Our main result is the following

Theorem 1. Let A be a finite-dimensional k-algebra. If the unbounded derived module category $D^{-}(\mathrm{Mod}-A)$ admits a symmetric recollement relative to the unbounded derived module categories of two finite-dimensional k-algebras B and C

$$D^{-}(\mathrm{Mod}-B) \overset{\leftarrow}{\rightleftharpoons} D^{-}(\mathrm{Mod}-A) \overset{\leftarrow}{\rightleftharpoons} D^{-}(\mathrm{Mod}-C),$$

then the unbounded derived module category $D^{-}(\mathrm{Mod}-T(A))$ admits symmetric recollement relative to the unbounded derived module categories of $T(B)$ and $T(C)$

$$D^{-}(\mathrm{Mod}-T(B)) \overset{\leftarrow}{\rightleftharpoons} D^{-}(\mathrm{Mod}-T(A)) \overset{\leftarrow}{\rightleftharpoons} D^{-}(\mathrm{Mod}-T(C)).$$

Before we end this section, we recall the concepts of total complexes of $\mathrm{Hom}^{\cdot}(X^{\cdot}, Y^{\cdot})$ and $X^{\cdot} \otimes Y^{\cdot}$ (see ref. [11]).

Given two complexes

$$X^{\cdot} = \cdots \longrightarrow X^{n-1} \xrightarrow{d_X^{n-1}} X^{n} \xrightarrow{d_X^{n}} X^{n+1} \longrightarrow \cdots$$

and

$$Y^{\cdot} = \cdots \longrightarrow Y^{n-1} \xrightarrow{d_Y^{n-1}} Y^{n} \xrightarrow{d_Y^{n}} Y^{n+1} \longrightarrow \cdots,$$

the total complex $\mathrm{Hom}^{\cdot}(X^{\cdot}, Y^{\cdot})$ is the complex given by

$$\mathrm{Hom}^{\cdot}(X^{\cdot}, Y^{\cdot}) := \cdots \longrightarrow \prod_{p\in\mathbb{Z}} \mathrm{Hom}(X^{p}, Y^{l+p}) \xrightarrow{d^{l}_{\mathrm{Hom}}} \prod_{p\in\mathbb{Z}} \mathrm{Hom}(X^{p}, Y^{l+1+p}) \longrightarrow \cdots,$$

where $d^{l}_{\mathrm{Hom}} : (\alpha^{p})_{p\in\mathbb{Z}} \mapsto (d_{Y}^{l+p}\alpha^{p} - (-1)^{l}\alpha^{p+1}d_{X}^{p})_{p\in\mathbb{Z}}$, for $\alpha_{p} \in \mathrm{Hom}(X^{p}, Y^{l+p})$, and the total complex $X^{\cdot} \otimes Y^{\cdot}$ is the complex given by

$$X^{\cdot} \otimes Y^{\cdot} := \cdots \longrightarrow \oplus_{p\in\mathbb{Z}} X^{p} \otimes Y^{l-p} \xrightarrow{d^{l}_{\otimes}} \oplus_{p\in\mathbb{Z}} X^{p} \otimes Y^{l+1-p} \longrightarrow \cdots,$$

where $d^l_{\otimes} : (x \otimes y)_{p\in\mathbb{Z}} \mapsto (d^p_X(x) \otimes y + (-1)^p x \otimes d^{l-p}_Y(y))_{p\in\mathbb{Z}}$, for $x \in X^p$, $y \in Y^{l-p}$.

2 The proof of main result

Based on Lemma A, we shall give a proof of the main theorem by a series of lemmas.

Lemma 1. Let $M^{\cdot} = \cdots \longrightarrow M^{n-1} \xrightarrow{d^{n-1}_M} M^n \longrightarrow \cdots$ be a bounded complex of projective A-modules, i.e. $M^{\cdot} \in K^b(\mathrm{Proj} - A)$. Then $R^{\cdot} = M^{\cdot} \otimes_A T(A)$ is a bounded complex of projective $T(A)$-modules, that is, $R^{\cdot} \in K^b(\mathrm{Proj} - T(A))$. Moreover, if $M^{\cdot} \in K^b(P_A)$, then we have $R^{\cdot} \in K^b(P_{T(A)})$.

Proof. For any $n \in \mathbf{Z}$, we have

$$(R^{\cdot})^n = (M^{\cdot} \otimes_A T(A))^n = M^n \otimes_A T(A),$$

but $M^{\cdot}$ is a bounded complex, so is $R^{\cdot}$.

Since any M^n is a projective A-module, supposing that $M^n = \varepsilon_n A$, where ε_n is an idempotent element of A, $M^n \otimes_{T(A)} T(A)= \varepsilon_n A \otimes_A T(A) \cong \varepsilon_n T(A)$ is a projective $T(A)$-module.

Moreover, if any M^n is a finitely generated A-module, it is easy to know that $(R^{\cdot})^n$ is also a finitely generated $T(A)$-module.

Lemma 2. Let $M^{\cdot} = \cdots \longrightarrow M^{n-1} \xrightarrow{d^{n-1}_M} M^n \longrightarrow \cdots$ be a partial tilting complex of A. Then $R^{\cdot} = M^{\cdot} \otimes_A T(A)$ is a partial tilting complex of $T(A)$.

Proof. First, we prove that $\mathrm{Hom}_{T(A)}(R^{\cdot}, R^{\cdot}[n]) = 0$ for any $n \neq 0$.

According to Iversen[11], the homology in degree n of total complex $\mathrm{Hom}^{\cdot}(R^{\cdot}, R^{\cdot})$ is naturally isomorphic to $\mathrm{Hom}_{T(A)}(R^{\cdot}, R^{\cdot}[n])$, so we only want to check $H^n(\mathrm{Hom}^{\cdot}(R^{\cdot}, R^{\cdot})) = 0$ for any $n \neq 0$.

According to the definition of total complex and properties of the category of complexes, and

$$\nu M^{\cdot} = D\mathrm{Hom}_A(M^{\cdot}, A) = \mathrm{Hom}_k(\mathrm{Hom}_A(M^{\cdot}, A), k) \cong M^{\cdot} \otimes_A \mathrm{Hom}_k(A, k) = M^{\cdot} \otimes_A DA,$$

where $\nu = D\mathrm{Hom}(-, A)$ is the Nakayama functor, we have

$$\begin{aligned}
\mathrm{Hom}^{\cdot}_{T(A)}(R^{\cdot}, R^{\cdot}) &= \mathrm{Hom}^{\cdot}_{T(A)}(M^{\cdot} \otimes_A T(A), M^{\cdot} \otimes_A T(A)) \\
&\cong \mathrm{Hom}^{\cdot}_A(M^{\cdot}, \mathrm{Hom}^{\cdot}_{T(A)}(T(A), M^{\cdot} \otimes_A T(A))) \\
&\cong \mathrm{Hom}^{\cdot}_A(M^{\cdot}, M^{\cdot} \otimes_A T(A)) \\
&\cong \mathrm{Hom}^{\cdot}_A(M^{\cdot}, M^{\cdot} \otimes_A (A \oplus D(A))) \\
&\cong \mathrm{Hom}^{\cdot}_A(M^{\cdot}, M^{\cdot}) \oplus \mathrm{Hom}^{\cdot}_A(M^{\cdot}, M^{\cdot} \otimes D(A)) \\
&\cong \mathrm{Hom}^{\cdot}_A(M^{\cdot}, M^{\cdot}) \oplus \mathrm{Hom}^{\cdot}_A(M^{\cdot}, \nu M^{\cdot}) \\
&\cong \mathrm{Hom}^{\cdot}_A(M^{\cdot}, M^{\cdot}) \oplus D\mathrm{Hom}^{\cdot}_A(M^{\cdot}, M^{\cdot}),
\end{aligned}$$

hence for any $n \neq 0$,

$$H^n(\mathrm{Hom}^{\cdot}_{T(A)}(R^{\cdot}, R^{\cdot})) \cong H^n(\mathrm{Hom}^{\cdot}_A(M^{\cdot}, M^{\cdot})) \oplus H^n(D\mathrm{Hom}^{\cdot}_A(M^{\cdot}, M^{\cdot})).$$

Because $M^{\cdot}$ is a partial tilting complex, for any $n \neq 0$, we have $H^n(\mathrm{Hom}^{\cdot}_A(M^{\cdot}, M^{\cdot})) = 0$, implying that $H^n(\mathrm{Hom}^{\cdot}_{T(A)}(R^{\cdot}, R^{\cdot})) = 0$. Therefore for any $n \neq 0$, $\mathrm{Hom}_{T(A)}(R^{\cdot}, R^{\cdot}[n]) = 0$.

Second, we prove that $\oplus_{i\in I}\mathrm{Hom}_{T(A)}(R^{\cdot}, R^{\cdot}) \overset{\text{nat.}}{\cong} \mathrm{Hom}_{T(A)}(R^{\cdot}, \oplus_{i\in I} R^{\cdot}_i)$, where $R^{\cdot}_i = R^{\cdot}$ for

$i \in I$. In fact,

$$\begin{aligned}\mathrm{Hom}_{T(A)}(R^{\cdot}, \oplus_{i\in I} R_i^{\cdot}) &\cong \mathrm{Hom}_{T(A)}(M^{\cdot} \otimes_A T(A), \oplus_{i\in I}(M_i^{\cdot} \otimes_A T(A)))\\ &\cong \mathrm{Hom}_{T(A)}(M^{\cdot} \otimes_A T(A), (\oplus_{i\in I} M_i^{\cdot}) \otimes_A T(A))\\ &\cong \mathrm{Hom}_A^{\cdot}(M^{\cdot}, \mathrm{Hom}_{T(A)}^{\cdot}(T(A), (\oplus_{i\in I} M_i^{\cdot}) \otimes_A T(A)))\\ &\cong \mathrm{Hom}_A^{\cdot}(M^{\cdot}, (\oplus_{i\in I} M_i^{\cdot}) \otimes_A T(A))\\ &\cong \mathrm{Hom}_A^{\cdot}(M^{\cdot}, \oplus_{i\in I} M_i^{\cdot}) \oplus \mathrm{Hom}_A^{\cdot}(M^{\cdot}, (\oplus_{i\in I} M_i^{\cdot}) \otimes_A DA)\\ &\cong \mathrm{Hom}_A^{\cdot}(M^{\cdot}, \oplus_{i\in I} M_i^{\cdot}) \oplus \mathrm{Hom}_A^{\cdot}(M^{\cdot}, \nu(\oplus_{i\in I} M_i^{\cdot})\\ &\cong \mathrm{Hom}_A^{\cdot}(M^{\cdot}, \oplus_{i\in I} M_i^{\cdot}) \oplus D\mathrm{Hom}_A^{\cdot}(\oplus_{i\in I} M_i^{\cdot}, M^{\cdot}))\\ &\cong \oplus_{i\in I} \mathrm{Hom}_A^{\cdot}(M^{\cdot}, M_i^{\cdot}) \oplus D\prod_{i\in I} \mathrm{Hom}_A^{\cdot}(M_i^{\cdot}, M^{\cdot})\end{aligned}$$

Because

$$\begin{aligned}D(\oplus_{i\in I}\mathrm{Hom}_A^{\cdot}(M_i^{\cdot}, M^{\cdot})) &\cong \mathrm{Hom}_k(\oplus_{i\in I} D\mathrm{Hom}_A^{\cdot}(M_i^{\cdot}, M^{\cdot}), k)\\ &\cong \prod_{i\in I} \mathrm{Hom}_k(D\mathrm{Hom}_A^{\cdot}(M_i^{\cdot}, M^{\cdot}), k)\\ &\cong \prod_{i\in I} \mathrm{Hom}_A^{\cdot}(M_i^{\cdot}, M^{\cdot}),\end{aligned}$$

we have $\oplus_{i\in I} D\mathrm{Hom}_A^{\cdot}(M_i^{\cdot}, M^{\cdot}) \cong D\prod_{i\in I} \mathrm{Hom}_A^{\cdot}(M_i^{\cdot}, M^{\cdot})$. Hence

$$\begin{aligned}&\mathrm{Hom}_{T(A)}(R^{\cdot}, \oplus_{i\in I} R_i^{\cdot})\\ \cong\ &\oplus_{i\in I}\ \mathrm{Hom}_A^{\cdot}(M^{\cdot}, M_i^{\cdot}) \oplus (\oplus_{i\in I} D\mathrm{Hom}_A^{\cdot}(M_i^{\cdot}, M^{\cdot}))\\ \cong\ &\oplus_{i\in I}\ (\mathrm{Hom}_A^{\cdot}(M^{\cdot}, M_i^{\cdot}) \oplus D\mathrm{Hom}_A^{\cdot}(M_i^{\cdot}, M^{\cdot}))\\ \cong\ &\oplus_{i\in I}\ (\mathrm{Hom}_A^{\cdot}(M^{\cdot}, M_i^{\cdot}) \oplus \mathrm{Hom}_A^{\cdot}(M^{\cdot}, \nu M_i^{\cdot}))\\ \cong\ &\oplus_{i\in I}\ (\mathrm{Hom}_A^{\cdot}(M^{\cdot}, M_i^{\cdot}) \oplus \mathrm{Hom}_A^{\cdot}(M^{\cdot}, M_i^{\cdot} \otimes_A DA))\\ \cong\ &\oplus_{i\in I}\ \mathrm{Hom}_A^{\cdot}(M^{\cdot}, M_i^{\cdot} \otimes_A T(A))\\ \cong\ &\oplus_{i\in I}\ \mathrm{Hom}_{T(A)}(M^{\cdot} \otimes_A T(A), M_i^{\cdot} \otimes_A T(A))\\ \cong\ &\oplus_{i\in I}\ \mathrm{Hom}_{T(A)}(R^{\cdot}, R_i^{\cdot}).\end{aligned}$$

Lemma 3. Assume that $M^{\cdot} \in K^b(\mathrm{Proj}-A)$ (or $M^{\cdot} \in K^b(P_A)$), $\mathrm{End}_A(M^{\cdot}) \cong B$. Set $R^{\cdot} = M^{\cdot} \otimes_A T(A)$. Then $\mathrm{End}_{T(A)}(R^{\cdot}) \cong T(B)$.

Proof. The proof is similar to that of Theorem 3.1 in ref. [3] or Lemma 3 in ref. [12].

Lemma 4. Assume that $M^{\cdot} \in K^b(\mathrm{Proj}-A)$, $N^{\cdot} \in K^b(P_A)$. Set $R^{\cdot} = M^{\cdot} \otimes_A T(A)$, $T^{\cdot} = N^{\cdot} \otimes_A T(A)$. If $\mathrm{Hom}_A^{\cdot}(N^{\cdot}, M^{\cdot}) = \mathrm{Hom}_A^{\cdot}(M^{\cdot}, N^{\cdot}) = 0$, then $\mathrm{Hom}_{T(A)}^{\cdot}(T^{\cdot}, R^{\cdot}) = \mathrm{Hom}_{T(A)}^{\cdot}(R^{\cdot}, T^{\cdot}) = 0$.

Proof. Since

$$\begin{aligned}&\mathrm{Hom}_{T(A)}^{\cdot}(T^{\cdot}, R^{\cdot})\\ =&\mathrm{Hom}_{T(A)}^{\cdot}(N^{\cdot} \otimes_A T(A), M^{\cdot} \otimes_A T(A))\\ \cong&\mathrm{Hom}_A^{\cdot}(N^{\cdot}, \mathrm{Hom}_{T(A)}^{\cdot}(T(A), M^{\cdot} \otimes_A T(A)))\\ \cong&\mathrm{Hom}_A^{\cdot}(N^{\cdot}, M^{\cdot} \otimes_A T(A))\end{aligned}$$

$$\cong \mathrm{Hom}^{\cdot}_A(N^{\cdot}, M^{\cdot}) \oplus \mathrm{Hom}^{\cdot}_A(N^{\cdot}, M^{\cdot} \otimes_A D(A))$$
$$\cong \mathrm{Hom}^{\cdot}_A(N^{\cdot}, \nu(M^{\cdot})) \quad (\text{becauce}\mathrm{Hom}^{\cdot}_A(N^{\cdot}, M^{\cdot}) = 0)$$
$$\cong D\mathrm{Hom}^{\cdot}_A(M^{\cdot}, N^{\cdot}),$$

we have $\mathrm{Hom}^{\cdot}_{T(A)}(T^{\cdot}, R^{\cdot})) = 0$.

In the same way we can show the second equality.

Lemma 5. Let $M^{\cdot}, N^{\cdot}, R^{\cdot}$ and $T^{\cdot}$ be as in Lemma 4. If $(M^{\cdot})^{\perp} \cap (N^{\cdot})^{\perp} = \{0\}$, then $(R^{\cdot})^{\perp} \cap (T^{\cdot})^{\perp} = 0$.

Proof. For any $X^{\cdot} \in (R^{\cdot})^{\perp} \cap (T^{\cdot})^{\perp}$, we have $\mathrm{Hom}^{\cdot}_{T(A)}(R^{\cdot}, X^{\cdot}[n]) = 0$ and $\mathrm{Hom}^{\cdot}_{T(A)}(T^{\cdot}, X^{\cdot}[n]) = 0$, for $n \in \mathbb{Z}$. Since

$$\begin{aligned}\mathrm{Hom}^{\cdot}_{T(A)}(R^{\cdot}, X^{\cdot}[n]) &= \mathrm{Hom}^{\cdot}_{T(A)}(M^{\cdot} \otimes_A T(A), X^{\cdot}[n]) \\ &\cong \mathrm{Hom}^{\cdot}_A(M^{\cdot}, \mathrm{Hom}^{\cdot}_{T(A)}(T(A), X^{\cdot}[n])) \\ &\cong \mathrm{Hom}^{\cdot}_A(M^{\cdot}, X^{\cdot}[n]),\end{aligned}$$

we have $\mathrm{Hom}^{\cdot}_A(M^{\cdot}, X^{\cdot}[n]) = 0$, thus $X^{\cdot} \in (M^{\cdot})^{\perp}$.

In the same way, we have $X^{\cdot} \in (N^{\cdot})^{\perp}$, so $X^{\cdot} = 0$. Therefore $(R^{\cdot})^{\perp} \cap (T^{\cdot})^{\perp} = 0$.

3 Remarks

Quasi-hereditary algebras are stratification algebras; that is, their derived module category admits recollement. Indeed, quasi-hereditary algebras have the nice property that their derived module category allows a stratification with subsequent quotients isomorphic to the derived module category $D^b(k)$ of the ground field k. We point out that strafication algebras are not necessarily quasi-hereditary algebras. In order to understand the examples of algebras whose derived category admits recollement, but which are not quasi-hereditary algebras, we show first

Theorem 2. Let A, B be a finite-dimensional algebras over an arbitrary base field k, ${}_AM_B$ be an A-B-bimodule, and as a k-vector space M be a finite dimension. Let $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be the triangle matrix algebra. If $pd(M_B) < \infty$, then

$$D^b(A) \overset{\leftarrow}{\rightleftharpoons} D^b(\Lambda) \overset{\leftarrow}{\rightleftharpoons} D^b(B).$$

Proof. Let $J = \begin{pmatrix} 0 & M \\ 0 & B \end{pmatrix}$. Then J is a two-sided ideal of Λ, and $A \cong \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \cong \Lambda/J$ (as k-algebras). ${}_\Lambda A \cong \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} = \Lambda \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$ is an idempotent element of Λ. Thus ${}_\Lambda A$ is a projective module. By Theorem 3.1 of ref. [8] or Theorem of ref. [10], there exists a triangulated category D'' such that $D^b(\Lambda)$ admits a recollement

$$D^b(A) \overset{\leftarrow}{\rightleftharpoons} D^b(\Lambda) \overset{\leftarrow}{\rightleftharpoons} D''.$$

In addition, $e = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}$ is also an idempotent element of Λ, and $e\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \cong B$,

$e\Lambda e = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \cong B$, so ${}_{e\Lambda e}e\Lambda$ is projective, hence $pd_{e\Lambda e}({}_{e\Lambda e}e\Lambda) < \infty$.

On the other hand, $\Lambda e = \begin{pmatrix} 0 & M \\ 0 & B \end{pmatrix}$; thus $\Lambda e_{e\Lambda e} \cong M_B \oplus B_B$. This implies $pd(\Lambda e_{e\Lambda e}) < \infty$. According to Theorem 3.1 of ref. [8], there exists a triangulated category D' such that $D^b(\Lambda)$ admits a recollement

$$D' \overset{\leftarrow}{\rightleftharpoons} D^b(\Lambda) \overset{\leftarrow}{\rightleftharpoons} D^b(B).$$

But $\Lambda e\Lambda = \begin{pmatrix} 0 & M \\ 0 & B \end{pmatrix}$, so $\Lambda/\Lambda e\Lambda \cong A$. By Theorem of ref. [10], we have $D' \cong D^b(A)$. This completes the proof of the theorem.

Collorary 1. Let A, B be finite-dimensional algebras over an arbitrary base field k, $\Lambda = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Then

$$D^b(A) \overset{\leftarrow}{\rightleftharpoons} D^b(\Lambda) \overset{\leftarrow}{\rightleftharpoons} D^b(B).$$

Collorary 2. Let A be a finite-dimensional algebra over an arbitrary base field k, and M be a left A-module. We denote $A[M] = \begin{pmatrix} A & M \\ 0 & k \end{pmatrix}$ the one-point extension of A by M. Then

$$D^b(A) \overset{\leftarrow}{\rightleftharpoons} D^b(A[M]) \overset{\leftarrow}{\rightleftharpoons} D^b(k).$$

In particular, if the derived module category $D^b(A)$ of A allows successively a stratification with subsequent quotients all of the simple form $D^b(k)$, then so is the derived module category $D^b(A[M])$ of $A[M]$.

The following examples show that stratification algebras are not necessarily quasi-hereditary algebras.

Example 1. By $M_n(k)$ we denote the algebra of $n \times n$ matrices over k. Let

$$\Lambda = \{\begin{pmatrix} a & 0 & 0 & e \\ 0 & b & 0 & 0 \\ 0 & d & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \in M_4(k) | a, b, c, d, e \in k\}.$$

Then $\Lambda \cong k\vec{Q}/I$, where the quiver $\vec{Q}$ and the relation I are given by

$$\vec{Q} = \ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \ \text{ and } \ I = \langle \alpha\beta \rangle.$$

Given the order $1 < 2 < 3$, clearly, ${}_\Lambda\Lambda$ is not a quasi-hereditary algebra. But if we set $A = k$, $B = \left\{ \begin{pmatrix} b & 0 & 0 \\ d & c & 0 \\ 0 & 0 & c \end{pmatrix} \in M_3(k) | b, c, d \in k \right\}$, $M = \{(0, 0, e) | e \in k\}$, then we have $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ and $pdM_B < \infty$. So by Theorem 2, $D^b(\Lambda)$ is stratifiable with the quotients $D^b(A)$ and $D^b(B)$.

Example 2. Let A be a finite-dimensional algebra over arbitrary base field k, and $gldA = \infty$. Set $\Lambda = \begin{pmatrix} A & {}_AA_A \\ 0 & A \end{pmatrix}$. Then by Theorem 2, we have $D^b(A) \overset{\leftarrow}{\rightleftharpoons} D^b(\Lambda) \overset{\leftarrow}{\rightleftharpoons} D^b(A)$; that is,

$D^b(\Lambda)$ is stratifiable with the quotients $D^b(A)$ and $D^b(A)$. But Λ is not a quasi-hereditary algebra for any given order.

In ref. [12] Du introduced a class of extension algebras that generalize the trivial extension algebra of A more extensively. This class of the extension algebras will be denoted by R_A^m. That is, if A is a finite-dimensional algebra over arbitrary base field k, then

$$R_A^m = \left\{ \begin{pmatrix} \lambda_1 & x_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & x_2 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_m & x_m \\ 0 & 0 & 0 & \cdots & 0 & \lambda_1 \end{pmatrix} | \lambda_i \in A, x_i \in D(A), 1 \leqslant i \leqslant m \right\}.$$

Clearly, $R_A^1 = T(A)$ is a trivial extension algebra of A, and $R_A^m = \hat{A}/\langle \nu^m \rangle$, where $\hat{A}$ is the repetitive algebra of A and ν is the Nakayama automorphism of $\hat{A}$. Similar to the proof of the main theorem, we have

Theorem 3. Let A be a finite-dimensional algbra, if the unbounded derived module category $D^-(\mathrm{Mod}-A)$ admits symmetric recollement relative to the unbounded derived module categories of two finite-dimensional algebras B and C:

$$D^-(\mathrm{Mod}-B) \overset{\leftarrow}{\rightleftarrows} D^-(\mathrm{Mod}-A) \overset{\leftarrow}{\rightleftarrows} D^-(\mathrm{Mod}-C),$$

then the unbounded derived category $D^-(\mathrm{Mod}-R_A^m)$ admits symmetric recollement relative to the unbounded derived module categories of R_B^m and R_C^m:

$$D^-(\mathrm{Mod}-R_B^m) \overset{\leftarrow}{\rightleftarrows} D^-(\mathrm{Mod}-R_A^m) \overset{\leftarrow}{\rightleftarrows} D^-(\mathrm{Mod}-R_C^m).$$

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References

1. Happel, D., Triangulated categories in the representation theories of finite dimensional algebra, London Lecture Notes Series 119, New-York: Cambridge University Press, 1988.
2. Rickard, J., Morita theory for derived categories, J. London Math. Soc., 1989, 39(2): 436—456.
3. Rickard, J., Derived categories and stable equivalence, J. Pure and Appl. Algebra, 1989, 61: 303—317.
4. Hughes, D., Waschbuesch, J., Trivial extensions of tilted algebras, Proc. London Math. Soc., 1982, 46(3): 347—364.
5. Grothendieck, A., Groups and classes des categories abeliennes et trianguliers complexe parfaits, LNM589, New York: Springer-Verlag, 1977. 351—371.
6. Beilinson, A. A., Bernstein, J., Deligne, P., Faisceaux pervers, in Analyse et topologie sur les espaces singuliers, Asterisque, 1982, 100: 1—172.
7. Cline, E., Parshall, B., Scott, L., Finite dimensional algebras and hight weightest categories, J. Reine Angew Math, 1988, 391: 85—99.
8. Cline, E., Parshall, B., Scott, L., Algebraic stratification in representative categories, J. of Algebra, 1988,117: 504—521.
9. Koenig, S., Tilting complexes, perpendicular categories and recollements of derived module categories of rings, J. Pure and Appl. Algebra, 1991, 73: 211—232.
10. Wiedemann, A., On stratificatios of derived module categories, preprint.
11. Iversen, B., Cohomology of Sheaves, Berlin: Springer-Verlag, 1986.
12. Du, X., Derived categories of algebra, Science in China, Series A, 1996, 26(11): 1002—1008.